Prediction of Stagnation and Minimum Pressure Points for Thin Films of Power Law and Bingham Liquids

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This study concerns the determination of stagnation point and minimum pressure point film thicknesses when a vertical flat plate is withdrawn from a reservoir containing non-Newtonian liquids. Upon consideration of three types of liquids, namely, Ellis, power law, and Bingham liquids, it has been found that stagnation point or minimum pressure point film thickness is a function of two parameters, one characterizing the liquid and the other representing parallel flow film thickness. It has also been shown that in the case of power law liquids, both the stagnation point and minimum pressure point exist, while in the case of Bingham liquids, the existence of one or both the points for a given nondimensional parallel flow film thickness depends upon the value of the Bingham number. Furthermore, it has been found that there may be situations when both the points can coincide; for example, in the case of Newtonian liquids, the two points coincide if the parallel flow film thickness is $\sqrt{2-\sqrt{3}}$.

When a solid object is withdrawn from a reservoir containing a liquid, there is a stagnation point in the thin liquid film adhering to the object which divides the regions of upward and downward flow within the film, at least for Newtonian liquids. Groenveld (1970*a*) and Lee and Tallmadge (1972*a*, *b*, 1973) have investigated this phenomenon and have developed methods for determining the thickness of the film at the stagnation point for Newtonian liquids. Similarly, the problem of determining minimum pressure film thickness was addressed by Lee and Tallmadge (1974). However, the literature reveals no attempts to determine stagnation and minimum pressure point film thicknesses for non-Newtonian liquids, though theoretical and experimental investigations on other flow characteristics for non-Newtonian films have been performed by Gutfinger and Tallmadge (1965), Tallmadge (1966*a*, *b*), Groenveld (1970*b*), and Denson (1972). It is, therefore, the purpose of this research to predict the two types of film thicknesses for Bingham and power law liquids and to find the condition on parallel flow film thickness in situations when the stagnation and minimum pressure points coincide.

Thus, if we take the x axis vertically upwards and y axis perpendicular to and originating at the plane of the flat wall and making the assumptions used by Gutfinger and Tallmadge (1965), the equation of motion becomes

$$\frac{1}{\rho}\frac{\partial \tau_{xy}}{\partial y} - g - \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \tag{1}$$

with the boundary conditions

$$u = u_w \quad \text{at} \quad y = 0 \quad \text{and} \quad \tau_{xy} = 0 \quad \text{at} \quad y = h \tag{2}$$

Let us first consider the Ellis liquid for which the constitutive equation is given by

$$\frac{\partial u}{\partial y} = \tau_{xy}\,[a + b\,|\tau_{xy}|^{r-1}] \tag{3}$$

The velocity profile and flow rate are given by

$$u = u_w + aC\left(\frac{y^2}{2} - hy\right) + \frac{bC^r}{r+1}\,[(h-y)^{r+1} - h^{r+1}] \tag{4}$$

and

$$Q_h = u_w h - \frac{aCh^3}{3} - \frac{bC^r\,h^{r+2}}{r+2} \tag{5}$$

respectively, where

$$C = \rho g + \frac{\partial p}{\partial x} \tag{6}$$

Since for the parallel flow region $\partial p/\partial x = 0$ and the film thickness is constant, say h_o, the corresponding flow rate Q_o can be determined from Equation (5) by replacing h and C by h_o and ρg, respectively. On using $Q_h = Q_o$, one then gets

$$u_w(h - h_o) + \frac{a\rho g h_o^3}{3} + \frac{b(\rho g)^r}{r+2}\,h_o^{r+2} = \frac{aCh^3}{3} + \frac{bC^r}{r+2}\,h^{r+2} \tag{7}$$

Again, one obtains the expression for surface velocity, u_s, by putting $y = h$ in Equation (4). Using this in conjunction with Equation (7) to eliminate C, we can then get the stagnation film thickness by setting $u_s = 0$, and hence we obtain

$$L_{st} = \frac{3}{1-r}\,F(L_{st}, T, T') + \frac{6^r\,T'^{(r+1)}[F(L_{st}, T, T')]^r}{L_{st}^{2r-2}\,(1-r)^r(1+r)\,T^{2r}} \tag{8}$$

where

$$F(L_{st}, T, T') = L_{st} - (r+2) + \frac{T^2(r+2)}{3} + T'^{(r+1)} \tag{9}$$

$$T = h_o\left(\frac{a\rho g}{u_w}\right)^{1/2} \tag{10}$$

and

$$T' = h_o\left[\frac{b(\rho g)^r}{u_w}\right]^{\frac{1}{r+1}} \tag{11}$$

To obtain the location of the minimum pressure point, we put $\partial p/\partial x = 0$ in Equation (7), which yields

$$L_m - 1 - \frac{T^2}{3}(L_m^3 - 1) - \frac{T'^{(r+1)}}{r+2}(L_m^{r+2} - 1) = 0 \tag{12}$$

We shall discuss below the two special cases of Equation (3), namely, $a = 1/\mu_o$, $b = -\tau_o/\mu_o$, $r = 0$ for $|\tau_{xy}| > \tau_o$, and $a = b = 0$ for $|\tau_{xy}| \leq \tau_o$, yielding

$$\begin{aligned} \frac{\partial u}{\partial y} &= \frac{\tau_{xy} + \tau_o}{\mu_o} \quad (|\tau_{xy}| > \tau_o) \\ &= 0 \qquad\quad (|\tau_{xy}| \leq \tau_o) \end{aligned} \tag{13}$$

which defines Bingham liquids, and $a = 0$, $b = k^{-1/n}$, $r = 1/n$ leading to

$$\tau_{xy} = k \left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial u}{\partial y} \tag{14}$$

which defines power law liquids.

BINGHAM LIQUIDS

For these liquids it may be observed from (1) and (2) that

$$\tau_{xy} = -C\,(h - y) \tag{15}$$

which leads to

$$C\,(h - y) > \tau_o \tag{16}$$

as a condition for flow.

Furthermore, condition (16) implies that there is a yield surface $y_1 = h - \tau_o/C$, and the region of flow is given by

$$0 \leq y \leq h - \frac{\tau_o}{C} = y_1 \tag{17}$$

where the flow, on using (13), (15), and (2), is described by

$$\begin{aligned} u &= u_w + \frac{C}{\mu_o}\left(\frac{y^2}{2} - hy\right) + \frac{\tau_o}{\mu_o} y \quad (0 \leq y \leq y_1) \\ u &= u_w + \frac{C}{\mu_o}\left(\frac{y_1^2}{2} - hy_1\right) + \frac{\tau_o}{\mu_o} y_1 \quad (y_1 \leq y \leq h) \end{aligned} \tag{18}$$

For simplicity, consider the case when the yield surface coincides with the free surface; that is, $\tau_o/C \to 0$. This can be true when τ_o, the yield stress, is small, and C is large. Thus, stagnation and minimum pressure film thicknesses are given by

$$B_i\, L^2_{st} - 2L_{st} + 3B_i - 2T_2^2 + 6 = 0 \tag{19}$$

and

$$2T_2^2\, L_m^2 + (2T_2^2 - 3B_i)L_m + 2T_2^2 - 3B_i - 6 = 0 \tag{20}$$

respectively, where

$$T_2 = h_o \left(\frac{\rho g}{\mu_o u_w}\right)^{1/2} \tag{21}$$

and

$$B_i = \frac{\tau_o h_o}{\mu_o u_w} \tag{22}$$

Since L_{st}, L_m, B_i, and T_2 are to be positive, Equations (19) and (20) give the results presented in Table 1 for the existence of stagnation points and minimum pressure points for Bingham liquids.

Now, as it is, the parameters B_i and T_2^2 for a given liquid with a given withdrawal velocity can be calculated only if the parallel flow film thickness h_o is known. Thus, once h_o is known and hence B_i and T_2^2, the existence of stagnation and minimum pressure points can be predicted by Table 1, and the corresponding film thicknesses can be obtained from Equations (19) and (20). Moreover, physically, it is observed from Table 1 that the existence of stagnation and minimum pressure points for a given liquid with a nonzero yield stress τ_o will depend upon the values of

$$\alpha = \frac{T_2^2 - 3 + \sqrt{(T_2^2 - 3)^2 + 3}}{3} \; \frac{\mu_o u_w}{h_o}$$

and

$$\beta = \frac{2(T_2^2 - 3)}{3} \; \frac{\mu_o u_w}{h_o}$$

where $T_2^2 = h_o^2 \rho g/\mu_o u_w$. For example, if $T_2^2 < 3$ and the yield stress is greater than α, there may be no stagnation point, while if $T_2^2 < 3$ and the yield stress has a value between β and α, there may be two stagnation points.

Further, assuming that there may be situations when the stagnation and minimum pressure points coincide, we get, on using $L_{st} = L_m$, from Equations (19) and (20)

$$B_i\, L^2_{st} - 2L_{st} + 3B_i - 2T_2^2 + 6 = 0 \tag{19}$$

and

$$2T_2^2\, L^2_{st} + (2T_2^2 - 3B_i)L_{st} + 2T_2^2 - 3B_i - 6 = 0 \tag{23}$$

On adding Equations (19) and (23), we get

$$L_{st} = \frac{3B_i - 2(T_2^2 - 1)}{B_i^2 + 2T_2^2} \tag{24}$$

which, on using either Equation (19) or (23), yields

$$4T_2^6 - 4(B_i + 4)T_2^4 + (B_i - 2)^2 T_2^2 - 6B_i^2(B_i + 1) = 0 \tag{25}$$

a cubic equation in T_2^2 with at least one positive root for a given B_i. However, rewriting Equation (25), we have

$$6B_i^3 - (T_2^2 - 6)B_i^2 + 4T_2^2(T_2^2 + 1)B_i - 4T_2^2(T_2^4 - 4T_2^2 + 1) = 0 \tag{26}$$

This equation does not give a positive value of B_i if

$$2 - \sqrt{3} \leq T_2^2 \leq 2 + \sqrt{3} \tag{27}$$

Also, Equation (24) shows that

$$B_i > \frac{2(T_2^2 - 1)}{3} \tag{28}$$

and combining this with

$$\frac{2(T_2^2 - 3)}{3} < B_i \leq \frac{T_2^2 - 3 + \sqrt{(T_2^2 - 3)^2 + 3}}{3} \tag{29}$$

there results

$$\frac{2(T_2^2 - 1)}{3} < B_i \leq \frac{T_2^2 - 3 + \sqrt{(T_2^2 - 3)^2 + 3}}{3} \tag{30}$$

to give the range for the value of B_i for a given T_2^2 if stagnation and minimum pressure points exist and coincide. Furthermore, Equation (30) gives

$$T_2^2 \leq \frac{11}{8} \tag{31}$$

Since Equation (26) does not have a positive root when $2 - \sqrt{3} \leq T_2^2 \leq 11/8$, it follows that for a value of

$$T_2^2 < 2 - \sqrt{3} \tag{32}$$

and B_i, the positive root of Equation (26), the stagnation

TABLE 1. CONDITIONS ON B_i FOR THE EXISTENCE OF STAGNATION AND MINIMUM PRESSURE POINTS FOR DIFFERENT T_2^2

T_2^2	B_i	Stagnation point	Minimum pressure point
$T_2^2 > 3$	$B_i \leq \dfrac{2(T_2^2 - 3)}{3}$	One	No
Any value	$\dfrac{2(T_2^2 - 3)}{3} < B_i < \dfrac{T_2^2 - 3 + \sqrt{(T_2^2 - 3)^2 + 3}}{3}$	Two	One
Any value	$B_i = \dfrac{T_2^2 - 3 + \sqrt{(T_2^2 - 3)^2 + 3}}{3}$	Two (coincident)	One
Any value	$B_i > \dfrac{T_2^2 - 3 + \sqrt{(T_2^2 - 3)^2 + 3}}{3}$	No	One

point, and the minimum pressure point will exist and coincide. For example, if $T_2^2 = 1/4$, Equation (26) gives $B_i = 1/24$ and $L_m = L_{st} = 3$. Similarly, if $T_2^2 = 1/8$, Equation (26) gives $B_i = 0.157514$ and $L_m = L_{st} = 5.453903$.

Again, if it is assumed that the dynamic meniscus film thickness is greater than the parallel flow film thickness, it follows that L_m and L_{st} should be greater than 1. This additional restriction then leads to the following: there is one and only one minimum pressure point if $B_i > T_2^2 - 1$ and no minimum pressure point if $B_i \leq T_2^2 - 1$, and, the number of stagnation points given in Table 1 indicates the maximum number of such points. For instance, if for a value of $T_2^2 > 3$, B_i is selected to satisfy $B_i \leq 2(T_2^2 - 3)/3$, then, according to Table 1 and Equation (19), there will exist one stagnation point if $B_i < 1$ and there may be one stagnation point or no stagnation point when $B_i > 1$, depending upon whether $\sqrt{1 - B_i(3B_i - 2T_2^2 + 6)}$ is greater than or less than $B_i - 1$. If L_m and L_{st} are equal, it is found that

$$2T_2^2 - 1 < B_i \leq \frac{T_2^2 - 3 + \sqrt{(T_2^2 - 3)^2 + 3}}{3}$$

which, on using (27), again leads to the result that stagnation and minimum pressure points coincide for values of B_i and T_2^2 satisfying $T_2^2 < 2 - \sqrt{3}$ and Equation (26).

POWER LAW LIQUIDS

Stagnation and minimum pressure film thicknesses for power law liquids are given by

$$L_{st} = \frac{2n + 1}{n} - T_1^{\frac{n+1}{n}} \tag{33}$$

and

$$L_m - 1 - \frac{n}{2n + 1} T_1^{\frac{n+1}{n}} \left(L_m^{\frac{2n+1}{n}} - 1 \right) = 0 \tag{34}$$

respectively, where

$$T_1 = h_o \left[\frac{1}{u_w} \left(\frac{\rho g}{k} \right)^{1/n} \right]^{n/(n+1)} \tag{35}$$

It has been shown by Gutfinger and Tallmadge (1965) that $T_1 \to 0$ when the capillary number Ca is very small and $T_1 \to 1$ when Ca is very large, showing that T_1 lies between 0 and 1. Now, for $0 < T_1 < 1$, it can be seen from Equation (33) that L_{st} is always greater than unity, whereas Equation (34) gives at least one value of L_m greater than unity. For such values of L_{st} and L_m and for a fixed T_1 and increasing n, L_{st} decreases while L_m increases. But for a fixed n and increasing T_1, both L_{st} and L_m decrease, with L_m remaining greater than L_{st} up to a certain value of T_1, beyond which L_m becomes less than L_{st}. Therefore, there is a value of T_1 between 0 and 1 for which stagnation and minimum pressure points coincide. Hence, on putting $L_m = L_{st}$ in Equations (33) and (34), we get

$$T_1^{\frac{n+1}{n}} \left[\frac{2n + 1}{n} - T_1^{\frac{n+1}{n}} \right]^{\frac{n+1}{n}} - \frac{n + 1}{n} = 0 \tag{36}$$

When $n = 1$, in which case the liquid is a Newtonian one, Equation (36) becomes

$$T_1^2(3 - T_1^2)^2 - 2 = 0 \tag{37}$$

which is cubic in T_1^2 and has the roots $\pm \sqrt{12}$, $\pm \sqrt{2 + \sqrt{3}}$, $\pm \sqrt{2 - \sqrt{3}}$, of which $T_1 = \sqrt{2 - \sqrt{3}} = 0.5176$ is permissible because $0 < T_1 < 1$. Hence stagnation and minimum pressure points for Newtonian liquids will coincide if the parallel flow film thickness is $\sqrt{2 - \sqrt{3}}$. When $T_1 < \sqrt{2 - \sqrt{3}}$, $L_{st} < L_m$, and when $T_1 > \sqrt{2 - \sqrt{3}}$, $L_{st} > L_m$. For values of n other than unity, we can get the values of T_1 from Equation (36), for which $L_{st} = L_m$. For example, in the case of pseudoplastic liquids with $n = 0.2$ and 0.4, the permissible values of T_1 from Equation (36) are obtained approximately as 0.1926 and 0.3192, respectively, and in the case of dilatant liquid with $n = 2$, the permissible value of T_1 is approximately 0.6725.

Finally, it may be noted from the above theoretical analysis using a one-dimensional model that the rheology of liquids affects very strikingly the stagnation and minimum pressure film thicknesses. Whereas Newtonian liquids always exhibit one stagnation point, non-Newtonian liquids may not have any stagnation point or may have more than one stagnation point.

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NOTATION

a, b = rheological constants, Equation (3)
Ca = capillary number $= u_w \mu / \sigma$
g = acceleration due to gravity

h, h_o = variable and constant film thickness
h_m, h_{st} = minimum pressure point and stagnation point film thickness
k = rheological constant, consistency index, Equation (14)
$L_m, L_{st} = h_m/h_o, h_{st}/h_o$
n = rheological constant, power law exponent, Equation (14)
p = pressure
Q_h, Q_o = flow rate in the variable and parallel flow region
r = rheological constant, Equation (3)
u, u_s, u_w = vertical, surface and withdrawal velocity
x, y = coordinates
y_1 = location of the yield surface

Greek Letters

μ = viscosity (Newtonian)
μ_o = viscosity (Bingham)
ρ = liquid density
σ = surface tension of the liquid-air interface
τ_o = yield stress
τ_{xy} = xy component of the stress tensor

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Stability of Chemical Reactors

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This note presents an extension of the studies of Chambré (1956) and Barkelew (1959) on the runaway criteria and parametric sensitivity of tubular nonisothermal chemical reactors. Chambré and Barkelew have developed an interesting method of obtaining graphical solutions of the nonlinear differential equation which describes the relation between temperature and degree of conversion in such reactors. The present note reports a simple procedure for determining the exact conditions leading to either stable or runaway reactor operation.

We consider, as an example, the case of an irreversible, first-order, exothermic reaction in the gas phase. Under steady state conditions, the relation between the temperature T and the degree of conversion X at any point in the tubular reactor is

$$\frac{dT}{dX} = \frac{(-\Delta H)y_{Ao}}{C_p} - \frac{4U(T - T_w)}{C_p DZP(1 - X)\exp(-E/RT)} \quad (1)$$

where the effect of radial temperature profiles is neglected.

Equation (1) can be written in the following dimensionless form

$$\frac{d\theta}{dX} = \alpha - \beta \frac{\theta \exp(-\theta)}{1 - X} \quad (2)$$

where

$$\theta = \frac{T - T_w}{T_w} \frac{E}{RT_w} \quad (3)$$

$$\alpha = \frac{(-\Delta H)y_{Ao}}{C_p T_w} \frac{E}{RT_w} \quad (4)$$

and

$$\beta = \frac{4U}{C_p DPk_w} \quad (5)$$

Chambré and Barkelew have shown that a graphical solution to Equation (2), which will be called the operating curve, can be obtained as follows. Let specific values be assigned to the parameters α and β as well as to the quotient $(d\theta/dX) = \theta_X$ in Equation (2), which can be written as

$$X = 1 - \frac{\beta\theta \exp(-\theta)}{\alpha - \theta_X} \quad (6)$$

Plots of θ vs. X in accordance with Equation (6) are shown in Figure 1 for a set of α and β values and various θ_X values. These plots have been called isoclines. The operating curve can now be drawn with the help of these isoclines and its initial and final slopes (Chambré, 1956):

1. The operating line originates at $\theta = 0$ and $X = 0$, and its initial slope is $\theta_X = \alpha$
2. At every intersection with an isocline, the operating curve must have a slope equal to the value of θ_X for that isocline.
3. The terminal slope of the operating curve at $X \to 1$ is $-\alpha/(\beta - 1)$.

Chambré and Barkelew have indicated the general areas of stable and unstable reactor operation on $\theta - X$ plots but have not determined the critical values of α